Pergamon

www.elsevier.com/locate/jappmathmech

PII: S0021–8928(04)00004-8

J. Appl. Maths Mechs, Vol. 68, No. 1, pp. 29–38, 2004

0021–8928/$—see front matter

THE SOLUTION OF MIXED PROBLEMS OF THE THEORY OF ELASTICITY FOR AN ANISOTROPIC HALF-PLANE AND COUPLED HALF-PLANES†

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(*Received* 4 *December* 2001)

The problem of the stretching of an elastic anisotropic plane with a bounded linear inclusion is solved. The problem is reduced to a Riemann boundary-value problem on two sheets of the complex plane glued together along the real axis. Computations are carried out for an inclusion situated at an angle to the anisotropy axis. © 2004 Elsevier Ltd. All rights reserved.

The method of boundary representations [1], developed to solve two-dimensional mixed problems of isotropic elasticity for a simply connected body, possesses several distinctive features and advantages.

1. Any component of the stress tensor (σ_n, σ_τ, τ_n) and strain tensor (ε_n, ε_τ, $\varepsilon_{n\tau}$), in "normal-tangent" coordinates (n, τ), and the derivative with respect to the arc coordinate of the Cartesian components of the displacement (u, υ) and the boundary of the body are expressed as linear combinations of the boundary values of a single set of functions, which are analytic in a disk or a half-plane, and of their complements to the entire plane. The same can be said of any linear combination of these objects (the unified mode of representation).

2. A set of boundary conditions on some part of the boundary is associated with a combination of boundary values of vectors of analytic functions with rational matrix coefficients. The absolute value of the determinant of the a matrix coefficient is identically equal to unity (the admissible class of domains comprises domains mapped conformally onto canonical domains by rational functions; uniformity of boundary conditions).

3. The totality of mixed boundary conditions is associated with a Riemann boundary-value problem on a circle or a straight line, with discontinuous matrix coefficients. Since the absolute value of the determinant of a matrix coefficient equals unity at every point of the boundary, only its argument undergoes a discontinuity. This implies that the boundary-value problem is unconditionally solvable (the model is mathematically well posed).

4. The modelling of a physical problem is a process that reduces to constructing a problem of the theory of functions from ready-made blocks that correspond to one or another type of boundary conditions. In the case of an unbounded body, the coupling condition must be supplemented by requirements that produce a specific stressed state in the remote zone (constructivity of the modelling).

5. In many cases the matrix coefficient can be factorized and a quadrature or explicit solution written out.

6. Thanks to the uniformity of the boundary representation one can consider, instead of a Riemann boundary-value problem, a system of singular integral equations for any pair of independent mechanical characteristics (flexibility of the apparatus).

7. In order to construct curves there is no need to reproduce the elastic field. The distribution of a mechanical quantity along the boundary may be reproduced directly in terms of the boundary values of the analytic functions found, using the formulae of its boundary representation (economy of means).

The extension of such an efficiency method to anisotropic objects would seem to be a timely topic.

As is well known [2], for two-dimensional anisotropy there are two possible ways to represent the field characteristics in terms of analytic functions. In the case of pairwise equal roots of the characteristic equation, the complex representation formulae are identical to the Kolosov–Muskhelishvili formulae, and therefore everything that has been done for isotropy is entirely repeatable [3]. In the case of pairwise-different roots, Lekhnitskii's complex representation expresses the field characteristics in terms of analytic functions of the variables relative to different complex planes, among whose points an affine

†*Prikl. Mat. Mekh.* Vol. 68, No. 1, pp. 35–44, 2004.

correspondence is established. Below, the apparatus of the method of boundary representations will be extended to bodies was straight boundaries, that is, half-planes or planes with singularities localized along a straight line or a set of segments and rays oriented along one straight line.

1. BOUNDARY REPRESENTATION FORMULAE FOR A HALF-PLANE

Suppose the complex half-plane $z = x + iy$, $y > 0$, is superposed on the plan of an elastic body. Lekhnitskii's complex representation formulae express the components of the stress tensor σ_x^+, σ_y^+, σ_{xy}^+ and strain tensor ε_x^+, ε_y^+, ε_{xy}^+, as well as the derivative of the displacement vector $u + iv$ with respect to the x coordinate, in terms of two functions $\Omega_1^+(z_1)$, $\Omega_2^+(z_2)$, each of which is analytic in its variable z_1, z_2 [2]:

$$\begin{aligned}
\sigma_x^+ &= 2\mathrm{Re}[\mu_1^2\Omega_1^+(z_1) + \mu_2^2\Omega_2^+(z_2)] \\
\sigma_y^+ &= 2\mathrm{Re}[\Omega_1^+(z_1) + \Omega_2^+(z_2)] \\
\sigma_{xy}^+ &= -2\mathrm{Re}[\mu_1\Omega_1^+(z_1) + \mu_2\Omega_2^+(z_2)] \\
\varepsilon_x^+ &= 2\mathrm{Re}[p_1\Omega_1^+(z_1) + p_2\Omega_2^+(z_2)] \\
\varepsilon_y^+ &= 2\mathrm{Re}[p_3\Omega_1^+(z_1) + p_4\Omega_2^+(z_2)] \\
\varepsilon_{xy}^+ &= -2\mathrm{Re}[p_5\Omega_1^+(z_1) + p_6\Omega_2^+(z_2)] \\
\partial v^+/\partial x &= 2\mathrm{Re}[q_1\Omega_1^+(z_1) + q_2\Omega_2^+(z_2)] + \omega \\
\omega_{xy} &= \frac{1}{2}(\partial u^+/\partial y - \partial v^+/\partial x) = \\
&= -2\mathrm{Re}[(q_1 - p_5)\Omega_1^+(z_1) + (q_2 - p_6)\Omega_2^+(z_2)] - \omega
\end{aligned} \tag{1.1}$$

where ω defines a rigid rotation, ω_{xy} is a rotation of an element of the medium, all constants are defined in terms of the constants of elasticity of anisotropy a_{ij}

$$\begin{aligned}
p_1 &= a_{11}\mu_1^2 + a_{12} - a_{16}\mu_1, \quad p_3 = \mu_1 q_1 \\
p_2 &= a_{11}\mu_2^2 + a_{12} - a_{16}\mu_2, \quad p_4 = \mu_2 q_2 \\
p_5 &= \mu_1(a_{11} - a_{12}), \quad p_6 = \mu_2(a_{11} - a_{12}) \\
q_1 &= a_{12}\mu_1 + a_{22}/\mu_1 - a_{26}, \quad q_2 = a_{12}\mu_2 + a_{22}/\mu_2 - a_{26}
\end{aligned}$$

The image of the physical plane is a pair of sheets $z_1 = x + \mu_1 y$, $z_2 = x + \mu_2 y$ bonded along their common axis $y = 0$. Any analytic function whose value on the x axis if $f(x)$ may be continued from this axis to each of the four half-planes either as $f(z_1)$ or as $f(z_2)$. Let D^+ denote the set of half-planes of z_1, z_2 for $y > 0$ and D^- their completion to a two-sheeted surface.

In terms of the functions $\Omega_1^+(z_1)$, $\Omega_2^+(z_2)$ we define two more functions that are analytic in half-planes identified by signs (using Muskhelishvili's notation [4])

$$\Omega_3^-(z_1) = \bar{\Omega}_1^+(z_1), \quad \Omega_4^-(z_2) = \bar{\Omega}_2^+(z_2) \tag{1.2}$$

The use of dual identification for these functions (the subscripts 1, 2 are here combined with the plus superscript, and 3, 4 are combined with the minus superscript) is superfluous, but it proves to be convenient in problems for a plane when, together with the field of the upper half-plane (the plus superscript), the field of the lower half-plane (the minus superscript) will also occur. After that, each of the functions is analytically continued to the other sheet through the common boundary. For the representative point of the two-sheeted surface we introduce the common notation z, setting $z = z_1$ or $z = z_2$ depending on the numbering of the sheets. Accordingly, after the analytic continuations have been carried out, each of the functions will depend on a single argument.

All the fundamental mechanical quantities will be expressed in terms of the boundary values of these functions at $z = z_1 = z_2 = x$. They are listed below in Table 1 (the boundary value of a mechanical quantity

Table 1

Mechanical quantities	Coefficients of the functions ($n = 1, 2$)	
	Ω_n^+	Ω_{n+2}^-
σ_x^+	μ_n^2	$\bar{\mu}_n^{2}$
σ_y^+	1	1
σ_{xy}^+	$-\mu_n$	$-\bar{\mu}_n$
$\varepsilon_x^+ = \partial u^+/\partial x$	p_n	$\bar{p}_n$
ε_y^+	p_{n+2}	$\bar{p}_{n+2}$
$2\varepsilon_{xy}^+$	p_{n+4}	$\bar{p}_{n+4}$
$\partial v^+/\partial x$	q_n	$\bar{q}_n$

in the first column equals a linear combination of the boundary values of the functions in the table header, with the specific coefficients indicated.

We will present a few examples of the construction of the boundary relations for common versions of the boundary conditions.

Surface forces. The normal and shear stresses are defined on the boundary. The rows for the stresses $p = \sigma_y^+$, $\tau = \sigma_{xy}^+$ give coupling conditions of the form

$$\begin{Vmatrix} \Omega_1^+ \\ \Omega_2^+ \end{Vmatrix} = G \begin{Vmatrix} \Omega_3^- \\ \Omega_4^- \end{Vmatrix} + g$$

$$G = \frac{1}{\mu_1 - \mu_2} \begin{Vmatrix} \mu_2 - \bar{\mu}_1 & \mu_2 - \bar{\mu}_2 \\ \bar{\mu}_1 - \mu_1 & \bar{\mu}_2 - \mu_1 \end{Vmatrix}, \quad g = \frac{1}{\mu_1 - \mu_2} \begin{Vmatrix} -\mu_2 p - \tau \\ \mu_1 p + \tau \end{Vmatrix} \tag{1.3}$$

Surface displacements. The displacements are defined on the boundaries. Apart from constants, the rows corresponding to these conditions are $u' = \partial u^+/\partial x$, $\upsilon' = \partial \upsilon^+/\partial x$ and the coefficients are

$$G = \frac{1}{q_2 p_1 - q_1 p_2} \begin{Vmatrix} \bar{q}_1 p_2 - q_2 \bar{p}_1 & \bar{q}_2 p_2 - q_2 \bar{p}_2 \\ q_1 \bar{p}_1 - \bar{q}_1 p_1 & q_1 \bar{p}_2 - \bar{q}_2 p_1 \end{Vmatrix}$$

$$g = \frac{1}{q_2 p_1 - q_1 p_2} \begin{Vmatrix} q_2 u' - p_2 \upsilon' \\ -q_1 u' + p_1 \upsilon' \end{Vmatrix}$$

Contact with a small profile. Under a rigid punch, the normal component of the displacement and the shear stress are defined (the later is equal to zero). The coefficients are

$$G = \frac{1}{q_2 \mu_1 - q_1 \mu_2} \begin{Vmatrix} \bar{q}_1 \mu_2 - q_1 \bar{\mu}_2 & \bar{q}_2 \mu_2 - q_2 \bar{\mu}_2 \\ q_1 \bar{\mu}_1 - \bar{q}_1 \mu_1 & q_1 \bar{\mu}_2 - \bar{q}_2 \mu_1 \end{Vmatrix}$$

$$g = \frac{1}{q_2 \mu_1 - q_1 \mu_2} \begin{Vmatrix} -q_2 \tau - \mu_2 \upsilon' \\ q_1 \tau + \mu_1 \upsilon' \end{Vmatrix}$$

The assumption that the stresses are finite in the far zone characterizes all the functions as bounded at infinity

$$\begin{aligned} \Omega_1^+(z_1) &= \Gamma + A z_1^{-1} + O(z_1^{-2}) \\ \Omega_2^+(z_2) &= \Gamma' + B z_2^{-1} + O(z_2^{-2}) \end{aligned} \tag{1.4}$$

If the parameters of the homogeneous stressed state and rotation in the remote zone are denoted by σ_x^∞, σ_y^∞, σ_{xy}^∞, ω^∞, the following relation is observed between them and the values of the functions at infinity, $\Gamma = \Omega_1^+(\infty)$, $\Gamma' = \Omega_2^+(\infty)$ as follows from expression (1.1)

$$\begin{Vmatrix} \sigma_x^\infty \\ \sigma_y^\infty \\ \sigma_{xy}^\infty \\ -2\omega^\infty \end{Vmatrix} = H \begin{Vmatrix} \mathrm{Re}\Gamma \\ i\mathrm{Im}\Gamma \\ \mathrm{Re}\Gamma' \\ i\mathrm{Im}\Gamma' \end{Vmatrix} \tag{1.5}$$

where

$$H = \begin{Vmatrix} \mu_1^2+\bar\mu_1^2 & \mu_1^2-\bar\mu_1^2 & \mu_2^2+\bar\mu_2^2 & \mu_2^2-\bar\mu_2^2 \\ 2 & 0 & 2 & 0 \\ -\mu_1-\bar\mu_1 & -\mu_1+\bar\mu_1 & -\mu_2-\bar\mu_2 & -\mu_2+\bar\mu_2 \\ \mathrm{Re}r_1 & i\mathrm{Im}r_1 & \mathrm{Re}r_2 & i\mathrm{Im}r_2 \end{Vmatrix}$$

$$r_1 = q_1 + p_5, \quad r_2 = q_2 + p_6$$

In each specified case, inversion of relations (1.5) enables the values of the functions at infinity to be determined. Lekhnitskii [1] formulated a uniquely solvable system of linear equations for the constants A and B in expansion (1.4)

$$\begin{Vmatrix} 1 & 1 & -1 & -1 \\ \mu_1 & \mu_2 & -\bar\mu_1 & -\bar\mu_2 \\ \mu_1^2 & \mu_2^2 & -\bar\mu_1^2 & -\bar\mu_2^2 \\ \mu_1^{-1} & \mu_2^{-1} & -\bar\mu_1^{-1} & -\bar\mu_2^{-1} \end{Vmatrix} \cdot \begin{Vmatrix} A \\ B \\ \bar A \\ \bar B \end{Vmatrix} = i \begin{Vmatrix} -\tilde Y \\ \tilde X \\ (a_{16}\tilde X + a_{12}\tilde Y)/a_{11} \\ -(a_{12}\tilde X + a_{26}\tilde Y)/a_{22} \end{Vmatrix} \tag{1.6}$$

$$\tilde X = X/(2\pi h), \quad \tilde Y = Y/(2\pi h)$$

$X + iY$ is the principal vector of forces applied to a plate of thickness h in the finite part of the plane.

2. SOLUTION OF THE COUPLING PROBLEM ON A TWO-SHEETED SURFACE

Let us consider boundary conditions "of the Nth kind" [1], when the mechanical quantities satisfy an appropriate set of different linear relations along the boundary of the body. Under such conditions the boundary of the body may be divided into N classes

$$\partial D^+ = l_1 \cup l_2 \cup \dots l_N, \quad l_k \cap l_j = \varnothing, \quad k \neq j$$

The construction of a boundary relation for the analytic vectors

$$\Omega^+ = \{\Omega_1^+, \Omega_2^+\}, \quad \Omega^- = \{\Omega_3^-, \Omega_4^-\}$$

reduces to the following boundary-value problem

$$\Omega^+ = G\Omega^- + g, \quad x \in \partial D^+; \quad G = G_n, \quad g = g_n, \quad x \in l_n \tag{2.1}$$

The matrix coefficient G is piecewise-constant. Defining $\Omega^- = \bar{\Omega}^+$ "by symmetry," we obtain the following two relations after changing to the conjugate in (2.1)

$$G\bar{G} = E, \quad G\bar{g} + g = 0$$

The first of these implies that $|\det G| = 1$.

It is required to find functions that are analytic on the surface obtained by bonding the two sheets along the real axis. These functions must satisfy symmetry condition (1.2) and restriction (1.3).

The problem may be reduced to an analogous problem on a plane. To that end, it suffices to ignore one of the sheets. Indeed, solution of the "truncated" problem for each of the sheets independently, with the solutions subsequently bonded, guarantees satisfaction of all the boundary conditions and the conditions at infinity. The symmetry of the functions, which holds within each single sheet, will be maintained after they have been bonded as well, since the index of the variable z "attaching" it to the sheet is "forgotten."

The main difficulty in solving the "truncated" problem is to factorize the matrix coefficient, that is, to represent it in the form

$$G = X^+(x)[X^-(x)]^{-1}, \quad x \in \partial D^+$$

where the function matrices (canonical solutions) participating in the product are the boundary values of matrices that are analytic in the sign-identified half-spaces and admit of a definite order at infinity [5]. The first factorization of the piecewise-constant matrix coefficient was achieved by Plemelj [6].

The general solution of the Riemann matrix boundary-value problem in the plane has the form

$$\Omega^\pm(z) = X^\pm(z)\left[\frac{1}{2\pi i}\int\limits_{\partial D^+}\frac{X^\pm(t)}{t-z}dt + P(z)\right], \quad z \in D^\pm \tag{2.2}$$

where $P(z)$ is a vector of polynomials with undetermined coefficients, guaranteeing the required behaviour of the unknown functions at infinity. The values of its coefficients are established by picking out a particular solution that guarantees not only the given stress level in the remote zone and the stipulated symmetry of the functions, but also special features due to the formulation of the problem (for example, the constants "lost" when the displacement is differentiated with respect to the boundary parameter must be reconstructed).

Remark. In cases in which the matrix coefficient of problem (2.1) has a diagonal structure, the boundary-value problem is reduced, and instead of a vector of functions it proves sufficient to consider a scalar function. The other functions are either defined by symmetry or are not needed.

3. BOUNDARY REPRESENTATION FORMULAE FOR COUPLED HALF-PLANES

Two anisotropic half-planes are coupled along a straight line, which is identified with the real axis of the complex plane. The state of the upper half-plane will be described using the functions, $\Omega_1^+(z_1)$, $\Omega_2^+(z_2)$ of formulae (1.1), in which the constants of elasticity are labelled with a plus superscript, and the state of the lower plane will be described using functions $\Omega_1^-(z_1)$, $\Omega_2^-(z_2)$ of the analogous formulae, taking a minus superscript for the constants of elasticity. For each of these functions, a symmetric function is defined in the coupled half-plane by formulae of the form (1.2). Thus we have defined two analytic vectors on both sheets of a two-sheeted surface, glued together along their common x axis, with the same complex variable

$$\Omega^\pm(z) = \{\Omega_1^\pm(z), \Omega_2^\pm(z), \Omega_3^\pm(z), \Omega_4^\pm(z)\}$$

In terms of the boundary values of this vector one can express not only the mechanical quantities listed in Table 1 but also the analogous quantities as viewed from the lower half-plane and their jumps of the form $[a] = a^+ - a^-$. These jumps are listed in Table 2.

Table 2

Discontinuities of the mechanical quantities	Coefficients of the functions ($n = 1, 2$)	
	$\Omega_n^{\pm}$	$\Omega_{n+2}^{\pm}$
$[\sigma_x]$	$\pm\mu_n^{\pm 2}$	$\mp\bar{\mu}_n^{\mp 2}$
$[\sigma_y]$	± 1	∓ 1
$[\sigma_{xy}]$	$\mp\mu_n^{\pm}$	$\pm\bar{\mu}_n^{\mp}$
$[\varepsilon_x] = \partial[u]/\partial x$	$\pm p_n^{\pm}$	$\mp\bar{p}_n^{\mp}$
$[\varepsilon_y]$	$\pm p_{n+2}^{\pm}$	$\mp\bar{p}_{n+2}^{\mp}$
$2[\varepsilon_{xy}]$	$\pm p_{n+4}^{\pm}$	$\mp\bar{p}_{n+4}^{\mp}$
$\partial[v]/\partial x$	$\pm q_n^{\pm}$	$\mp\bar{q}_n^{\mp}$

The procedure by which the boundary relations for the analytic functions are constructed does not differ in principle from that described previously. We will present a few examples of typical relations at the places where the half-planes are coupled together, assuming that the latter are of a homogenous material (and therefore the plus and minus superscripts are omitted).

The condition of continuity of the field. In the case when the stresses and strains undergo no discontinuities along the line of coupling, the coefficients of the boundary-value problem take the simplest possible values.

$$G = E, \quad g = 0 \tag{3.1}$$

A rigid linear inclusion. A homogeneous inclusion moves like an absolutely rigid body, so that displacements are defined on it as viewed from each of the half-planes (apart from three real constants). To determine the coefficients of the coupling condition, one can retain the expressions for the displacement components from Table 1 and their discontinuities from Table 2. We obtain the expressions

$$G = \begin{Vmatrix} 0 & 0 & \alpha/\delta & \beta/\delta \\ 0 & 0 & \gamma/\delta & \bar{\alpha}/\delta \\ \bar{\alpha}/\bar{\delta} & -\beta/\bar{\delta} & 0 & 0 \\ -\gamma/\bar{\delta} & \alpha/\bar{\delta} & 0 & 0 \end{Vmatrix}, \quad g = \omega_0 \begin{Vmatrix} 0 \\ 1 \\ 0 \\ 1 \end{Vmatrix} \tag{3.2}$$

$$\delta = q_2 p_1 - q_1 p_2, \quad \alpha = q_2 \bar{p}_1 - \bar{q}_1 p_2$$
$$\beta = q_2 \bar{p}_2 - \bar{q}_2 p_2, \quad \gamma = \bar{q}_1 p_1 - q_1 \bar{p}_1$$

in which ω_0 is the angle of rotation.

A cut. Along a cut whose sides do not interact, the normal and shear stresses vanish. Retaining the corresponding rows in Table 1 (twice, since for the body occupying the lower half-plane one has the same representations but with a minus superscript) gives

$$G = \begin{Vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{Vmatrix}, \quad g = 0 \tag{3.3}$$

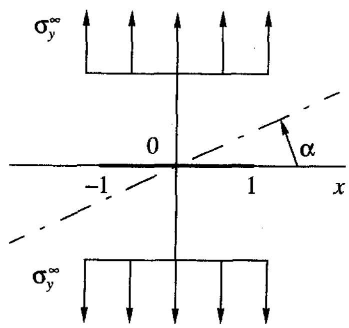

Fig. 1

When boundary conditions are formulated for four-dimensional vectors, cases may also arise in which reduction is admissible. Such cases are due to singularities of the structure in the common piecewise matrix coefficient, namely, the existence of diagonal blocks in the matrix.

4. UNIAXIAL STRETCHING OF A PLANE CONTAINING AN INCLUSION

Suppose an inclusion is oriented arbitrarily with respect to the anisotropy axes, at an angle α. Place the origin at its midpoint and assume that the x axis points along it. We shall assume that the inclusion occupies a closed interval $[-a, a]$ (Fig. 1). The following boundary conditions hold along the axis

$$\begin{aligned} &v^{\pm} = \omega_0 x, \quad \varepsilon_x^{\pm} = 0, \quad x \in l_1 \\ &[\sigma_y] = 0, \quad [\sigma_{xy}] = 0, \quad [u] = 0, \quad [v] = 0 \\ &x \in l_2 = (-\infty, -a) \cup (a, \infty) \end{aligned} \tag{4.1}$$

ω_0 corresponds to rigid rotation of the inclusion.

After application of the boundary representation formulae (Tables 1 and 2), one proceeds from (2.1) to a Riemann boundary-value problem on the common segment of a two-sheeted surface in the class of functions bounded at infinity

$$\Omega^+ = G\Omega^- + \omega_0 g, \quad x \in l_1 \tag{4.2}$$

The coefficients G and g are defined as in formulae (3.2).

Since the coefficient matrix in problem (4.2) is constant along an open contour, it may be factorized by first representing it in Jordan normal form and then changing to set of one-dimensional problems on the open contour [1]. The matrix G admits of a factorization

$$G = R\Lambda P^{-1}, \quad \Lambda = \mathrm{diag}\{1, 1, -1, -1\}$$

The constant elements of the matrix $R = [R_{ij}]$ are uniquely defined.

The change of variables $\Psi^{\pm}(z) = R^{-1}\Omega^{\pm}(z)$, where the argument belongs to either sheet, leads to the matrix form of a set of one-dimensional Riemann boundary-value problems on the segment

$$\Psi^+ = \Lambda\Psi^-, \quad x \in l_1 \tag{4.3}$$

Only the third and fourth rows of (4.3) have the coefficient -1, which is factorized by completing the segment to a straight line by a canonical function $\Pi(z) = 1/\sqrt{z^2 - a^2}$, which has different limits depending on whether the line l is approached from above or below; the remaining functions are continuous on l. Therefore the canonical matrix of problem (4.2) is defined as

$$X^{\pm}(z) = R\mathrm{diag}\{1, 1, \Pi(z), \Pi(z)\}$$

The general solution of problem (4.2), guaranteeing that the functions are bounded at infinity, is

$$\Omega^{\pm}(z) = X^{\pm}(z)\left\|\begin{matrix} P_1 \\ P_2 \\ P_3 + Q_3 z \\ P_4 + Q_4 z \end{matrix}\right\| \tag{4.4}$$

The fact that the principal vector vanishes and conditions (4.2) at infinity uniquely define all the undetermined coefficients of the general solution, thereby singling out a particular solution satisfying the mechanical meaning of the problem. In the case that the inclusion does not experience rotation, the coefficients are

$$\left\|\begin{matrix} P_1 \\ P_2 \\ Q_3 \\ Q_4 \end{matrix}\right\| = R^{-1}\left\|\begin{matrix} \Gamma \\ \Gamma' \\ \bar{\Gamma} \\ \bar{\Gamma}' \end{matrix}\right\|, \quad P_3 = P_4 = 0$$

The sequence of formulae (4.4), (1.1) enables the mechanical field to be constructed. Direct application of the boundary representation formulae yields curves of all the mechanical characteristics along the real axis without constructing the field.

Suppose real values are taken for the constants of elasticity of an orthotropic material [2]

$$E_1 = 2E_2 = 120\text{MPa}, \quad G = 7\text{MPa}, \quad \nu_1 = 2\nu_2 = 0.072$$

The coefficients expressed in terms of these parameters are

$$a_{11} = 1/E_1, \quad a_{22} = 1/E_2, \quad a_{12} = -\nu_1/E_1 = -\nu_2/E_2$$
$$a_{16} = a_{26} = 0, \quad a_{66} = 1/G$$

and the roots of the characteristic equation are $\mu_1 = 4.11i$, $\mu_2 = 0.343i$. The undetermined coefficients of the general solution at $\alpha = 0$ were taken to be

$$P_1 = -0.025, \quad P_2 = 0.71, \quad Q_3 = -3.934\times 10^{-4}, \quad Q_4 = 2.941\times 10^{-3},$$
$$P_3 = P_4 = 0$$

Computations were carried out for $\alpha = 0, 15°, 30°, 45°, 60°, 75°, 90°$. Stress curves σ_x, σ_y and σ_{xy} along the x axis, constructed directly by the representation formulae of Table 1, are shown in Figs 2 and 3 for the right semi axis as viewed from the upper half-plane. At $\alpha = 0$ the normal stress σ_y is uniformly distributed along the x axis, the stress σ_x is uniform within the limits of the inclusion (because of the constant stress σ_y and the absence of deformations in the long fibre), and on the boundary of the inclusion it is compressive and unbounded in absolute value. Shear stresses, on the contrary, are not present beyond the limits of the inclusion and unbounded near its end. The field characteristics of the stress–strain state may be constructed by analysing the analytic functions thus found and using Lekhnitskii's representation (1.1). As is obvious from the data presented in Fig. 2, when the anisotropy axes are rotated, the stresses stretching fibres contiguous with the inclusion become qualitatively different both within the inclusion and outside it. The stresses stretching fibres normal to the inclusion become qualitatively different only within the inclusion. The shear stresses become qualitatively different only beyond the limits of the inclusion.

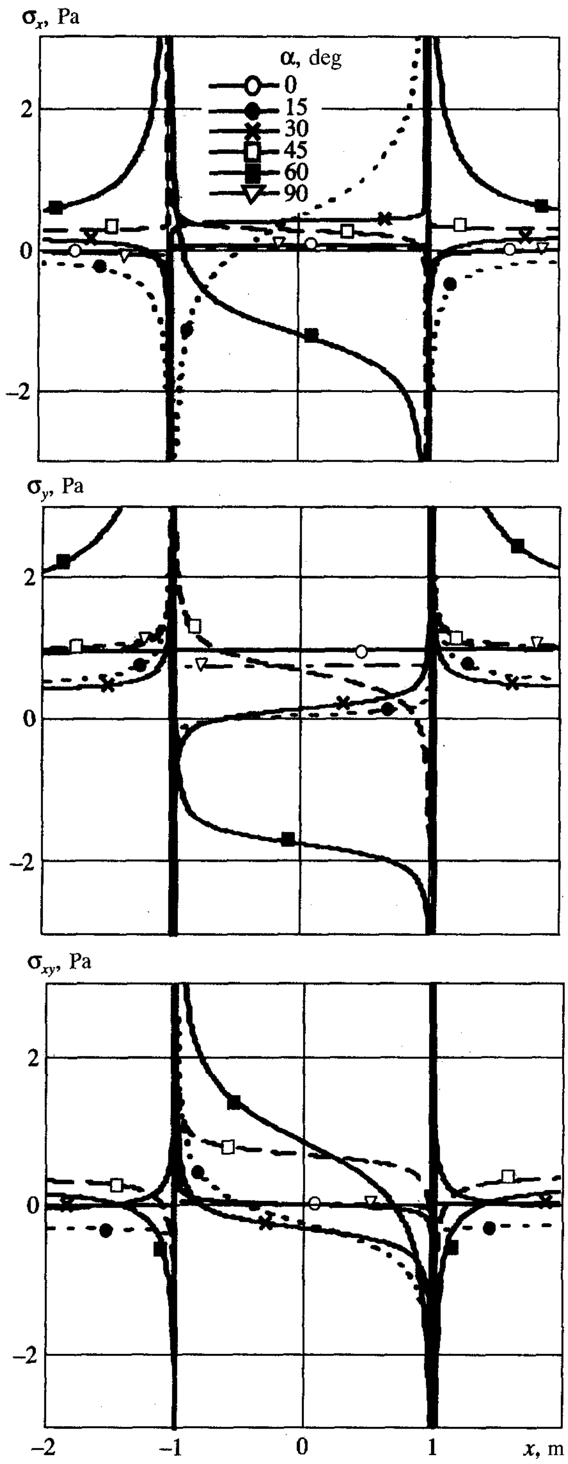
σ_x, Pa
α, deg
0
15
30
45
60
90
2
0
–2
σ_y, Pa
2
0
–2
σ_{xy}, Pa
2
0
–2
–2
–1
0
1
x, m

Fig. 2

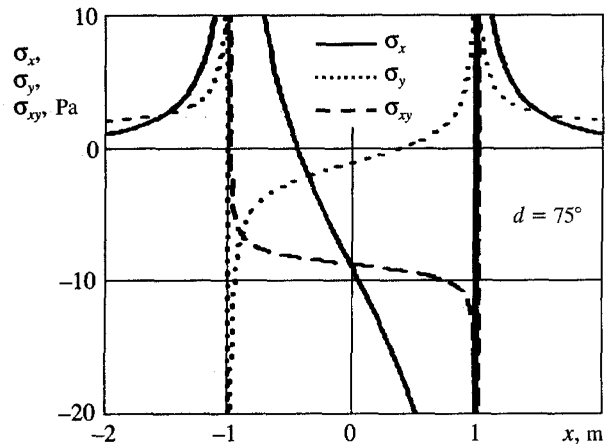

Fig. 3

5. CONCLUSIONS

The method proposed here enables exact solutions to be constructed in many mixed problems for bodies with straight boundaries. The solution may also be constructed numerically, using the "basic Riemann problem" apparatus [1], which reduces the boundary-value problem to a system of singular integral equations. In particular, it seems interesting to consider the following problems: the fundamental solution for an anisotropic medium containing a linear rigid inclusion, a system of collinear laminas at the juncture of heterogeneous anisotropic half-planes, the contact problem for an anisotropic half-plane, etc.

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Translated by D.L.